

Boundary conditions for integrable equations

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Received 9 August 1996, in final form 6 December 1996

Abstract. The problem of constructing boundary conditions for nonlinear equations compatible with higher symmetries is considered. In particular, this problem is discussed for the sine–Gordon, Jiber–Shabat, Liouville and KdV equations. New results are obtained for the last two ones. The boundary condition for the KdV contains two arbitrary constants. The substitution $u = q$, maps it onto the boundary condition with linear dependence on t for the potentiated KdV.

1. Introduction

Applications of classical Lie symmetries to boundary value problems are well known [1]. In contrast, the question of involving higher symmetries in the same problem has received much less attention, unlike, say, the Cauchy problem. However, it is now well understood that the approach of higher symmetries has become the basis of modern integrability theory [2]. A number of attempts to apply the inverse scattering method (ISM) to the initial boundary value problem have been made. It turned out that if both the initial data and boundary value are chosen arbitrarily, then the ISM essentially loses its power. On the other hand, the investigation by Sklyanin [3] based on the R-matrix approach demonstrated that there are certain boundary conditions that are completely compatible with integrability. The analytical aspects of such kind of problems were studied in [4]. It later became clear that the boundary value problems found can effectively be investigated with the help of the Backlund transformation [6].

Below we will discuss a higher symmetry test, proposed in [7, 8] to verify whether the boundary condition given is compatible with the integrability property of the equation. It is worth noting that all known classes of boundary conditions that are compatible with integrability pass this symmetry test. In [8–10] we gave an approach for finding the boundary conditions compatible with infinitely many higher symmetries. Boundary conditions involving explicit time dependence for the Toda lattice compatible with higher symmetries have recently been studied in [11]. It was observed that finite-dimensional systems obtained from the Toda lattice by imposing boundary conditions consistent with symmetries are closely connected with Painleve-type equations.

In previous works [8–10] we have mostly considered boundary conditions compatible with homogeneous symmetries. In this work we extend some of our previous results by utilising inhomogeneous symmetries. We show that for some integrable equations this

extension changes the boundary conditions found previously. As an example, the boundary conditions $u = \text{constant}$ and $u_{xx} = \text{constant}$ found in this work for the KdV equation

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(see the proposition in section 3) are new. Notice that these boundary conditions found in a formal algebraic way turned out to have nice analytic properties. Recently in [12] it was shown that the KdV equation admits regular finite-gap solutions with these boundary conditions.

2. Symmetry-compatible boundary conditions

Before presenting our results we give a brief review of our previous work [7, 8, 10]. Let an integrable nonlinear partial differential equation and a set of boundary conditions be given by

$$u_t = f(u, u_1, u_2, \dots, u_n) \quad (1) \quad p(u, u_1, u_2, \dots, u_k)|_{x=x_0} = 0 \quad (2)$$

where $p = (p_1, \dots, p_s)$ is a vector function, $s < n$. Here u_i stands for the partial derivative of order i with respect to the variable x . Suppose that equation (1) possesses a higher symmetry

$$u_\tau = g(u, u_1, \dots, u_m). \quad (3)$$

We call the problem (1), (2) compatible with the symmetry (3) if for any initial data prescribed at the point $t = 0$, a common solution to equations (1), (3) exists satisfying the boundary condition (2). Let us explain more exactly what we mean. Evidently, one can only differentiate the constraint (2) with respect to the variables t and τ (but not with respect to x). For instance, it follows from (2) that

$$\sum_{i=0}^n \frac{\partial p}{\partial u_i} (u_i)_\tau = 0 \quad (4)$$

where one should replace τ -derivatives by means of equation (3).

Definition. The boundary value problem (1), (2) is called compatible with the symmetry (3) if equation (4) holds identically by means of the condition (2) and its consequences obtained by differentiation with respect to t .

To formulate an effective criterion of compatibility of the boundary value problem with a symmetry it is necessary to introduce some new set of dynamical variables consisting of the vector $v = (u, u_1, u_2, \dots, u_{n-1})$ and its t -derivatives: v_t, v_{tt}, \dots . Passing to this set of variables allows one really to exclude the dependence on the variable x . In terms of these variables the symmetry (3) and the constraint (2) take the form

$$v_\tau = G \left(v, v_t, v_{tt}, \dots, \frac{\partial^{m_1} v}{\partial t^{m_1}} \right) \quad (5)$$

$$P \left(v, \frac{\partial v}{\partial t}, \dots, \frac{\partial^{k_1} v}{\partial t^{k_1}} \right) = 0 \quad (6)$$

The following criterion of compatibility was established in [8].

Theorem 1. The boundary value problem (1), (2) is compatible with the symmetry (3) if and only if the differential constraint (6) is consistent with the system (5).

We call the boundary condition (2) compatible with the integrability property of equation (1) if the problem (1), (2) is compatible with an infinite series of linearly independent higher-order symmetries.

The motivation for such a definition originated with the widely studied examples of boundary value problems for nonlinear equations (see [3, 4, 13]) consistent with the inverse scattering transform method.

Our main observation is that if the boundary condition is compatible with one higher symmetry then as a rule it is compatible with an infinite number of symmetries which form a set S with infinite elements. The set S may or may not contain the whole symmetries of (1). For instance, S contains the even-ordered time-independent symmetries for the Burgers equation.

We suppose that equation (1) admits a recursion operator (definition of integrability of (1)) of the form (see [2, 14, 15])

$$R = \sum_{i=0}^{i_1} \alpha_i D^i + \sum_{i=0}^{k_1} \alpha_{-1,i} D^{-1} \alpha_{-2,i} \quad i_1 > 0 \quad k_1 > 0 \quad (7)$$

where α_i , $\alpha_{-1,i}$, $\alpha_{-2,i}$ are functions of the dynamical variables, D is the total derivative with respect to x . Recursion operator when applied to a symmetry produces new symmetries. Passing to the new dynamical variables v, v_t, v_{tt}, \dots , one can obtain, from (7), the recursion operator \mathbf{R} of the system of equations (5):

$$\mathbf{R} = \sum_{i=0}^M \mathbf{a}_i (\partial_t)^i + \sum_{i=0}^K \mathbf{a}_{-1,i} (\partial_t^{-1}) \mathbf{a}_{-2,i} \quad M > 0 \quad K > 0 \quad (8)$$

where the coefficient matrices \mathbf{a}_i , $\mathbf{a}_{-1,i}$, $\mathbf{a}_{-2,i}$ depend on v and on a finite number of its t -derivatives, and ∂_t is the operator of the total derivative with respect to t . If (1) is a scalar equation, then R is a scalar operator and \mathbf{R} is an $n \times n$ matrix-valued operator. Our further considerations are based on the following propositions, which really affirm that if an equation admits a differential connection, then an infinite number of its higher symmetries admits also the same connection. These are propositions 2.2 and 2.1 of [10], respectively.

Theorem 2. Suppose that $p(v) = 0$ is set of $n-1$ functionally independent constraints and that there exists a positive integer n_0 such that the coefficient matrix b_N in the expression

$\mathbf{R} \text{ compatible with the symmetry } y_{n_0} = b_N (\partial_t)^N + b_{N-1} (\partial_t)^{N-1} + \dots + \sigma_t$ is proportional to the identity matrix. Then $\mathbf{R}^{n_0} \sigma_t$ if and only if it is compatible with the symmetry $p(v) = 0$ is

$\sigma_t = H(\mathbf{R}^{n_0}) \sigma_t$, where H is a scalar polynomial with constant coefficients, and $\sigma = v^T$.

This theorem plays a crucial role in determining the test symmetry $\mathbf{R}^{n_0} \sigma_t$ (or the positive integer n_0) which is going to be the first element of S . Boundary conditions found with the aid of this theorem will in general be compatible with only homogenous higher symmetries. Hence they will not be most general symmetry-compatible boundary conditions. For this purpose we have the following theorem.

Theorem 3. Suppose that $p(v) = 0$ is a set of $n - 1$ functionally independent constraints. If $p(v) = 0$ is compatible with the symmetry $\sigma_\tau = T(\mathbf{R})\sigma_t$ then it is compatible with every symmetry $\sigma_\tau = H(T(\mathbf{R}))\sigma_t$ where T and H are arbitrary scalar polynomials with constant coefficients.

The problem of the classification of integrable boundary conditions is solved completely for the Burgers equation (see [8])

$$u_t = u_2 + 2uu_1. \quad (9)$$

Theorem 4. If the boundary condition $p(u, u_1)|_{x=x_0} = 0$ is compatible at least with one higher symmetry of the Burgers equation (9) then it is compatible with all even-order homogeneous symmetries and is of the form $c_1(u_1 + u^2) + c_2u + c_3 = 0$.

3. The KdV equation

Very often the simplest integrable boundary condition can be found by the use of the reflection-type symmetry like $x \rightarrow -x$, $u \rightarrow h(u)$. It is unexpected that equations which do not admit any reflection symmetry nevertheless admit boundary conditions compatible with integrability. Let us consider, for instance, the famous KdV equation

$$u_t = u_{xxx} + 6uu_x. \quad (10)$$

As was established in [8], the KdV equation admits the homogeneous boundary condition

$$u = 0 \quad u_{xx} = 0 \quad x = x_0. \quad (11)$$

A further generalization of the above boundary condition is as follows:

$$u = c \quad u_{xx} = -3c^2 \quad x = x_0 \quad (12)$$

where c is an arbitrary constant. Note that the boundary conditions (11) and (12) consist of two independent scalar constraints and, as is proved in [8], are compatible with all homogeneous symmetries of orders divisible by three and also with every symmetry from the linear envelope of symmetries divisible by three (see below).

One of the principal problems here is to find the set S and to identify the simplest symmetries from S which can be considered as test symmetries. Due to the theorems 2 and 3, boundary conditions compatible with a test symmetry will be compatible with the all elements of the set S . It is evident, for instance, that to describe all boundary conditions for the Burgers equation consistent with integrability it is sufficient to examine only one test symmetry, namely the fourth-order homogeneous symmetry:

$$u_{\tau_2} = u_{xxxx} + 4u u_{xxx} + 10u_x u_{xx} + 6u^2 u_{xx} + 12u u_x^2 + 4u^3 u_x. \quad (13)$$

All boundary conditions compatible with this symmetry will be compatible with all even symmetries of the Burgers equation. In [11] it was shown that the Toda lattice is served by a whole family of test symmetries depending on a numerical parameter. So actually the set S may depend on the parameters.

Starting with this observation and with the boundary condition (12) for which the test symmetry coincides with the ninth-order homogeneous one, we now use theorem 3. As a test symmetry, in this case we take a linear combination of all symmetries of the KdV equation of both types either higher or classical with undefined coefficients. Thus we have seven summands: classical symmetries corresponding to Galilean invariance, x -translation, t -

translation, scaling and three higher-order symmetries, fifth, seventh and ninth. First let us give the ninth-order higher symmetry of the KdV equation:

$$u_{\tau_3} = f_9 = u_{ttt} + 96u^3 u_t + 54u^4 v + 18u^2 w_t + 36u^2 vw + 36uu_t w - 6u_t v^2 + 6ww_t + 6vw^2 \quad (14)$$

written in terms of the new dynamical variables $u, v = u_x, w = u_{xx}$ and their t -derivatives.

Now we consider the inhomogeneous symmetry

$$u_{\tau'_3} = f'_9 = f_9 + a_1 f_7 + a_2 f_5 + a_3 f_3 + a_4 f_1 + a_5 (\text{scale symmetry}) + a_6 (\text{Galilean symmetry}) \quad (15)$$

where f_j denotes the j th-order symmetry of the KdV equation, for each j , and in particular $f_1 = u_x = v$. Using $u_x = v$ and $u_{xx} = w$, we then get the following system of equations from (15) by differentiating it with respect to x :

$$\begin{aligned} u_{\tau_{30}} = u_{\tau_3} + a_1 (v_{tt} + 22u^2 u_t + 2uw_t + 4wu_t + 4uvw &+ 8u^3 v - 2v^3) + a_2 (w_t + 4uu_t + 2v(w + 3u^2)) + a_3 u_t + a_4 v \\ &+ a_5 (3tu_t + xv + 2u) + a_6 (1 + 6tv) v_{\tau_{30}} = \\ v_{\tau_3} + a_1 (w_{tt} + 2uu_{tt} + 2vw_t + 4v_t w + 12uvu_t &+ 10u^2 v_t + 4u^2_t + 8u^3 w + 4uw^2 - 2v^2 w) + a_2 (u_{tt} - 2uv_t \\ &+ 2w(w + 3u^2)) + a_3 v_t + a_4 w + a_5 (3tv_t + xw + 3v) + 6a_6 tw \end{aligned} \quad (16)$$

$$\begin{aligned} w_{\tau_{30}} = w_{\tau_3} + a_1 (u_{ttt} - 4uv_{tt} - 2vu_{tt} - 4uvv_t &- 2v^2 u_t + 6ww_t + 8u^3 u_t + 10u^2 w_t + 20uwu_t - 48u^4 v \\ &- 24u^2 vw + 12uv^3) + a_2 (v_{tt} - 2vv_t - 2uw_t + 4wu_t - 12uvw \\ &+ 6u^2 u_t - 36u^3 v) + a_3 w_t + a_4 (u_t - 6uv) + a_5 (3tw_t \\ &+ xu_t - 6xuv + 4w) + a_6 (6tu_t - 36tuv) \end{aligned}$$

where $u_{\tau_j} = f_{3+2j}$, $\forall j = -1, 0, 1, 2, \dots$, u_{τ_3} is given in (14) and $v_{\tau_3} = D_x u_{\tau_3}$, $w_{\tau_3} = D_x v_{\tau_3}$. The boundary conditions with one constraint are not compatible with the higher symmetries. Hence we now look for boundary conditions with two constraints $p_1(u, v, w) = 0$, $p_2(u, v, w) = 0$ compatible with the above system. Here we have the following distinct types of these constraints:

- (i) $u = h_1(w)$, $v = h_2(w)$,
- (ii) $v = h_1(u)$, $w = h_2(u)$,
- (iii) $u = h_1(v)$, $w = h_2(v)$.

Direct calculations lead to the following:

Proposition. The constraints

$$u = k_1 \quad u_{xx} = k_2 \quad x = x_0 \quad (17)$$

are integrable boundary conditions for the KdV equation where k_1 and k_2 are arbitrary constants and the non-zero constant coefficients in the system (16) are given by $a_2 = -6m$ and $a_4 = 6m^2$ with $m = 3k_1^2 + k_2$. The boundary conditions (17) are compatible with a set of infinitely many higher symmetries by theorem 3.

Note that the symmetry (15) contains the classical symmetry f_1 as a summand, and because of this one cannot use theorem 3 immediately. On the other hand some of the higher symmetries with which the boundary condition (17) are compatible, are, for instance:

$$u_{t60} = f_{15} - 10mf_{11} + 30m^2 f_7$$

and, applying theorem 3, the boundary conditions will be compatible with an infinite set of symmetries. Further calculations show that the above boundary conditions are also compatible with the following higher inhomogeneous symmetry containing the term f_1 :

$$u_{t90} = f_{21} - 14mf_{17} + 70m^2 f_{13} - 770m^4 f_5 + 868m^5 f_1.$$

The boundary conditions given in the above proposition are the most general ones. Previously [8–10] we had $k_1 = k_2 = 0$.

Now let us consider the potentiated KdV equation

$$q_t = q_{xxx} + 3q_x^2$$

which is connected to (10) by the formula $u = q_x$. It is easy to check that this substitution maps the boundary condition (17) onto the boundary condition with explicit dependence on t :

$$q = kt + c \quad q_x = a \quad x = x_0$$

the compatibility of which with integrability can also be checked directly. It is consistent with the symmetries

$$\begin{aligned} q_t &= f_3 - kf_{-1} & q_t &= f_9 - 6kf_5 + 6k^2 f_1 & q_t &= f_{15} - \\ & 10kf_{11} + 30k^2 f_7 - 25k^4 f_{-1} & & \dots \end{aligned}$$

where the $q_0 = f_j$ are the homogeneous symmetries rewritten in terms of the variable q and $f_{-1} = 1$.

Another example of boundary conditions with explicit t dependence can be obtained by the known point transformation which maps the KdV equation (10) onto the cylindrical KdV equation

$$u_t = u_{xxx} + 6uu_x - \frac{u}{2t}.$$

The boundary conditions take the form

$$u|_{x=x_0} = \frac{A}{t} \quad u_{xx}|_{x=x_0} = \frac{B}{t^2}$$

where A, B are arbitrary constants.

Finally, the integrable boundary condition for the mKdV equation

$$u_t = u_{xxx} + 6u^2u_x \quad (18)$$

$$u = 0 \quad u_x = a$$

is compatible with the symmetries

$$x = x_0$$

$$u_\tau = f_9 + 6a^2f_5 + 6a^4f_1 \quad u_\tau = f_{15} + 10a^2f_{11} + 30a^4f_7 \quad u_\tau = f_{21} +$$

$$14a^2f_{17} + 70a^4f_{13} - 770a^8f_5 - 868a^{10}f_1 \dots$$

Of course this result can be obtained by use of the Miura map.

4. Hyperbolic-type equations

The definition of the integrable boundary condition remains the same for hyperbolic-type equations. The only difference is that in this case the higher symmetry contains partial derivatives with respect to both independent variables t and x . Let us describe, for example, the boundary value problems of the form

$$p(u, u_x) = 0 \quad x = x_0 \quad (19) \quad u_{tt} - u_{xx} + \sin u = 0 \quad (20)$$

for the sine-Gordon equation, which are compatible with the third-order symmetry.

It was shown in [16] that the complete algebra of higher symmetries for this equation, i.e. $u_{\xi\eta} = \sin u$, where $2\xi = x + t$, $2\eta = x - t$ splits into the direct sum of two algebras consisting of the symmetries of the equations $u_\tau = u_{\xi\xi\xi} + u_\xi^3/2$, $u_\tau = u_{\eta\eta\eta} + u_\eta^3/2$, respectively, which are quite simply the potentiated mKdV equation. In particular, the following flow commutes with the sine-Gordon equation:

$$u_\tau = c_1(u_{\xi\xi\xi} + u_\xi^3/2) + c_2(u_{\eta\eta\eta} + u_\eta^3/2). \quad (21)$$

It can easily be verified that the symmetry (21) is compatible with the boundary condition of the form (19) only if constraint holds $c_1 = -c_2$. Under this constraint the symmetry is rewritten in original variables as the system

$$u_\tau = 8u_{ttt} + 6u_t \cos u + 3w^2u_t + u^3t \quad (22)$$

$$w_\tau = 8w_{ttt} + 6w_t \cos u + 6u_t w u_t + 3w^2w_t + 3u^2t w_t.$$

where $w = u_x$. According to the theorem 1 above one reduces the problem of finding integrable boundary conditions to the problem of looking for differential connections admissible by this system. Straightforward calculations prove that the boundary condition (19) has to have one of the forms

$$u = \text{constant} \quad u_x + a \cos(u/2) + b \sin(u/2) = 0 \quad x = x_0. \quad (23)$$

Note that the list of boundary conditions (23) coincides with that found by Zamolodchikov within the framework of the R-matrix approach [17]. The particular cases of (23) were studied earlier in [3, 5]. The compatibility of the first boundary condition in (23) with the usual version of ISM was declared earlier in [5]. However, the statement was based on a mistake (see [13]). Our requirement of consistency is weaker than that used in [5]. Applications of

these and similar problems for the sine–Gordon equation and the affine Toda lattice in the quantum field theory are studied in [18–20].

One can prove that the boundary conditions (23) are compatible with a rather large subclass of symmetries of the sine–Gordon equation such that

$$u_\tau = \varphi(u, u_1, \dots, u_{k_1}) - \varphi(u, u_1, \dots, u_{k_1})^- \quad (24)$$

where $u_j = \partial^j u / \partial \xi^j$, $u_j = \bar{\partial}^j u / \partial \eta^j$, and equation $u_\tau = \varphi_i(u, u_1, \dots, u_{k_i})$ is a symmetry of equation $u_\tau = u_{\xi\xi\xi} + u^3_\xi/2$ for $i = 1, 2$.

Another well known integrable hyperbolic-type equation

$$u_{tt} - u_{xx} = \exp(u) + \exp(-2u) \quad (25)$$

has applications in the geometry of surfaces. It was first found by Tzitzeica [21]. The presence of higher symmetries for this equation has been established by Jiber and Shabat [16]. The simplest higher symmetry of this equation is of fifth order:

$$u_\tau = u_{\xi\xi\xi\xi\xi} + 5(u_{\xi\xi}u_{\xi\xi\xi} - u_{2\xi}u_{\xi\xi\xi} - u_{\xi}u_{2\xi\xi}) + u_5\xi. \quad (26)$$

It was proved in [22] that the higher symmetry algebra for (25) is the direct sum of the higher symmetry algebras of (26) and of equation obtained from (26) by replacing ξ by η .

Let us look for boundary conditions of the form

$$p(u, u_x) = 0 \quad x = x_0 \quad (27)$$

for equation (25) that are compatible with the symmetry u_τ

$$\begin{aligned} &= u_{\xi\xi\xi\xi\xi} + 5(u_{\xi\xi}u_{\xi\xi\xi} - u_{2\xi}u_{\xi\xi\xi} - u_{\xi}u_{2\xi\xi}) + u_5\xi - u_{\eta\eta\eta\eta\eta} \\ &\quad - 5(u_{\eta\eta}u_{\eta\eta\eta} - u_{2\eta}u_{\eta\eta\eta} - u_{\eta}u_{2\eta\eta}) - u_5\eta. \end{aligned} \quad (28)$$

Rather simple but tediously long computations lead to the following statement [23].

Theorem 5. The boundary conditions (27) for the Jiber–Shabat equation compatible with the symmetry (28) (and then compatible with integrability) are either of the form

$$u_x + a \exp(-u)|_{x=x_0} = 0 \quad \text{or} \quad u_x + a \exp(u/2) \pm \exp(-u)|_{x=x_0} = 0$$

where a is arbitrary.

The famous Liouville equation u_{tt}

$$-u_{xx} + \exp(u) = 0$$

admits the rather rich symmetry algebra (see [16]). Let us take as a test symmetry the following simplest one:

$$u_\tau = 8u_{ttt} + u_t(6\exp(u) - 3u_x^2 - u_t^2).$$

Passing to the appropriate set of dynamical variables $u, w = u_x, u_t, w_t, \dots$, one obtains the system of the form

$$u_\tau = 8u_{ttt} + u_t(6\exp(u) - 3w^2 - u_t^2) \quad w_\tau = 8w_{ttt} +$$

$$w_t(6\exp(u) - 3w^2) - 3u_t(w_t u_t + 2w u_{tt})$$

where $w = u_x$. It is not difficult to find boundary conditions compatible with this symmetry:

$$u_x + a \exp(-u/2) + b \exp(u/2)|_{x=x_0} = 0$$

or

$$u|_{x=x_0} = \text{constant}.$$

5. Conclusion

In the search for symmetry-compatible boundary conditions [8–10] of the integrable nonlinear partial differential equations we had so far used homogeneous generalized symmetries. In this work we showed that one may obtain more general boundary conditions if inhomogeneous generalized symmetries are used.

We applied the technique developed in our previous work [8–10] to the Burgers, KdV, mKdV and hyperbolic-type equations by the use of inhomogeneous generalized symmetries. The boundary conditions obtained for Burgers remain the same. This confirms our uniqueness theorem for the boundary conditions of the Burgers equation [10]. The boundary conditions found here for the KdV and mKdV equations are new and more general than our previous results. In this paper we also included a new equation: the cylindrical KdV equation. The boundary conditions for this equation are time dependent. In the case of hyperbolic-type equations only the boundary conditions of the Liouville equation are new. The boundary conditions for others like sine–Gordon and Jiber–Shabat remain the same. The use of inhomogeneous generalized symmetries does not provide us with new boundary conditions.

Acknowledgments

This work has been supported by Russian Foundation for Fundamental Research and by the Turkish Scientific and Technical Research Council (TUBITAK). MG is an associate member of Turkish Academy of Sciences (TUBA).

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